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Differential geometry of the q-superplane

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Abstract. Hopf algebra structures on the extended q-superplane and its differential algebra are defined. An algebra of forms which are obtained from the generators of the extended q-superplane is introduced and its Hopf algebra structure is given.

1. Introduction

Differential geometry of Lie groups plays an important role in the mathematical modelling of physics theories. A class of noncommutative Hopf algebra has been found in the discussions of integrable systems. These Hopf algebras are q-deformed function algebras of classical groups and this structure is called a quantum group [1]. The quantum group can also be regarded as a generalization of the notion of a group [2]. Thus it is also attractive to generalize the corresponding notions of differential geometry. Mathematical aspects of such a generalization are promising. More recently it has been suggested that the zero branes in matrix theory [3] should be identified with supercoordinates in noncommutative geometry [4].

Noncommutative geometry [4] has started to play an important role in different fields of mathematical physics over the last few years. The basic structure giving a direction to the noncommutative geometry is a differential calculus on an associative algebra. The noncommutative differential geometry of quantum groups was introduced by Woronowicz in [5]. In this approach the quantum group is taken as the basic noncommutative space and the differential calculus on the group is deduced from the properties of the group. The other approach, initiated by Wess and Zumino [6], succeeded Manin's emphasis [7] on the quantum spaces as the primary objects, differential forms are defined in terms of noncommuting (quantum) coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces. The natural extension of their scheme to superspace [8] was introduced by Soni in [9].

The quantum superplane is the simplest example of a noncommutative superspace. We have investigated the noncommutative geometry of the quantum superplane. In section 2 we introduce two noncommutative differential calculi on the q-superplane. One of them is quite different from the calculus described in [9], where $GL_q(1|1)$ covariance was assumed. The graded Hopf algebra structures of the extended q-superplane and these supercalculi are given in section 3. In the following section we introduce two forms from the differential algebra and also give the graded Hopf algebra structure of the obtained algebra of forms.

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2. Differential calculi on the q-superplane

Let us begin with the Manin superplane. The quantum superplane is defined as an associative algebra whose even coordinate x and the odd (Grassmann) coordinate θ satisfy

$$x\theta - q\theta x = 0 \qquad \theta^2 = 0 \tag{1}$$

where q is a nonzero complex parameter. The algebra of q-polynomials will be called the algebra of functions on the quantum two-dimensional supervector space (superplane) and will be denoted by $\mathcal{A}$.

In order to establish a noncommutative differential calculus on the quantum superplane, we assume that the commutation relations between the coordinates and their differentials are in the following form:

$$\begin{aligned} x\,\mathrm{d}x &= A\mathrm{d}x\, x \\ x\,\mathrm{d}\theta &= F_{11}\mathrm{d}\theta\, x + F_{12}\mathrm{d}x\,\theta \\ \theta\,\mathrm{d}x &= F_{21}\mathrm{d}x\,\theta + F_{22}\mathrm{d}\theta\, x \\ \theta\,\mathrm{d}\theta &= B\mathrm{d}\theta\,\theta. \end{aligned} \tag{2}$$

The coefficients A, B and F_{ij} will be determined in terms of the complex deformation parameter q. To find them we shall use the consistency of calculus. We first note that the properties of the exterior differential. The exterior differential d is an operator which gives the mapping from the generators of $\mathcal{A}$ to the differentials

$$\mathrm{d} : u \longrightarrow \mathrm{d}u \qquad u \in \{x, \theta\}. \tag{3}$$

We demand that the exterior differential d has to satisfy two properties: nilpotency

$$\mathrm{d}^2 = 0 \tag{4}$$

and the graded Leibniz rule

$$\mathrm{d}(fg) = (\mathrm{d}f)g + (-1)^{\hat{f}} f(\mathrm{d}g) \tag{5}$$

where $\hat{f} = 0$ for even variables and $\hat{f} = 1$ for odd variables. From the consistency conditions

$$\mathrm{d}(x\theta - q\theta x) = 0 \qquad \mathrm{d}(\theta^2) = 0$$

we find

$$F_{11} + qF_{22} = q \qquad F_{12} + qF_{21} = -1 \qquad B = 1. \tag{6a}$$

Similarly, from

$$(x\theta - q\theta x)\,\mathrm{d}x = 0 \qquad (x\theta - q\theta x)\,\mathrm{d}\theta = 0$$

one has

$$F_{12}F_{22} = 0 \qquad (F_{11} - qA)F_{22} = 0. \tag{6b}$$

The system (6) has, at least, two solutions and we shall discuss them below.

We now define the commutation relations between variables and their differentials in the following form

$$Z^i\mathrm{d}Z^j = (-1)^{\hat{i}(\hat{j}+1)}C^{ji}{}_{kl}\mathrm{d}Z^k Z^l \tag{7}$$

where $C \in \mathrm{End}(\mathcal{C} \otimes \mathcal{C})$. Comparing (7) with (2) we obtain the general matrix C

$$C = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -F_{21} & -F_{22} & 0 \\ 0 & F_{12} & F_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

In the language of matrix C, associativity and consistency with the properties of d requires that C fulfil the following conditions:

$$C_{12}C_{13}C_{23} = C_{23}C_{13}C_{12} \qquad \hat{C}_{12}\hat{C}_{23}\hat{C}_{12} = \hat{C}_{23}\hat{C}_{12}\hat{C}_{23} \tag{9}$$

where $C_{12} = C \otimes I$, etc $\hat{C} = PC$ and P is the superpermutation matrix. The general matrix $\hat{C}$ may have, at least, one of two distinct forms:

$$\hat{C}_I = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & pq & 0 \\ 0 & q^{-1} & p-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{12} = 0 \tag{10a}$$

and

$$\hat{C}_{II} = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & r & q & 0 \\ 0 & qr-1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{22} = 0 \tag{10b}$$

where $p, r, s \in \mathcal{C}$ are free parameters. Similar matrices are found in [10] to obtain differential calculi on the quantum plane.

The matrix $\hat{C}_I$ satisfies all required conditions. If we set $s = qr$ then the matrix $\hat{C}_{II}$ also obeys all required conditions [11]. Each of these matrices leads to a family of differential calculi on the q-superplane.

So we have the following commutation relations. For $\hat{C}_I$

$$\begin{aligned} &x\,\mathrm{d}x = p\mathrm{d}x\,x \qquad x\,\mathrm{d}\theta = pq\mathrm{d}\theta\,x \\ &\theta\,\mathrm{d}x = -q^{-1}\mathrm{d}x\,\theta + (1-p)\mathrm{d}\theta\,x \qquad \theta\,\mathrm{d}\theta = \mathrm{d}\theta\,\theta. \end{aligned} \tag{11a}$$

For $\hat{C}_{II}$

$$\begin{aligned} &x\,\mathrm{d}x = s\mathrm{d}x\,x \qquad x\,\mathrm{d}\theta = q\mathrm{d}\theta\,x + (qr-1)\mathrm{d}x\,\theta \\ &\theta\,\mathrm{d}x = -r\mathrm{d}x\,\theta \qquad \theta\,\mathrm{d}\theta = \mathrm{d}\theta\,\theta. \end{aligned} \tag{11b}$$

In the case of family I, it is easy to check that the differential structure is invariant under action of quantum supergroup $GL_q(1|1)$ (see, e.g. [12]) if we take $p = q^{-2}$. Similarly one can see, in the case of family 2, that the differential structure is invariant under action of $GL_{q,r}(1|1)$ (see, e.g. [13]) if we set $s = qr$.

Applying the exterior differential d to the first and second (or third) relations of (11) we obtain

$$(\mathrm{d}x)^2 = 0 \qquad \mathrm{d}x\,\mathrm{d}\theta = pq\mathrm{d}\theta\,\mathrm{d}x \tag{12a}$$

for family I and

$$(\mathrm{d}x)^2 = 0 \qquad \mathrm{d}x\,\mathrm{d}\theta = r^{-1}\mathrm{d}\theta\mathrm{d}x \tag{12b}$$

for family II.

A differential algebra on an associative algebra $\mathcal{B}$ is a z_2-graded associative algebra Γ equipped with an operator d that has the properties (3)–(5). Furthermore, the algebra Γ has to be generated by $\Gamma^0 \cup \Gamma^1 \cup \Gamma^2$, where Γ^0 is isomorphic to $\mathcal{B}$. For $\mathcal{B}$ we write $\mathcal{A}$. Let us denote the algebra (as a matter of fact the module) generated by $\mathrm{d}x$ and $\mathrm{d}\theta$ with the relations (11) by Γ^1, where Γ^1 is isomorphic to $\mathrm{d}\mathcal{A}$, and the algebra (12) by Γ^2. Let Γ be the quotient algebra of the free associative algebra on the set $\{x, \theta, \mathrm{d}x, \mathrm{d}\theta\}$ modulo the ideal J that is generated by the relations (1), (11) and (12).

In section 3 we shall show that the algebra $\mathcal{A}$ (q-superplane), the algebra Γ^1 and also Γ^2 are all the graded Hopf algebras and so is the algebra Γ.

3. Hopf algebra structures

A Hopf algebra structure on the quantum plane was introduced in [14]. In this section we introduce a graded Hopf algebra structure on the algebra $\mathcal{A}$ (i.e. on the q-superplane) and give the natural extension on Γ.

3.1. A Hopf algebra structure on $\mathcal{A}$

We know, from section 1, that the quantum superplane, $\mathcal{A}$, is an associative algebra over a field k generated by two elements x, θ obeying the relations (1). We can now define a coproduct and a counit on the algebra $\mathcal{A}$ as follows.

The coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\begin{aligned} \Delta(x) &= x \otimes x \\ \Delta(\theta) &= \theta \otimes x + x \otimes \theta \\ \Delta(1) &= 1 \otimes 1. \end{aligned} \tag{13}$$

The counit $\epsilon : \mathcal{A} \longrightarrow \mathcal{C}$ is given by

$$\epsilon(x) = 1 \qquad \epsilon(\theta) = 0. \tag{14}$$

The algebra $\mathcal{A}$ with the coproduct and the counit has a structure of bi-algebra. One extends the algebra $\mathcal{A}$ by including inverse of x which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

If we extend the algebra $\mathcal{A}$ by adding the inverse of x then the algebra $\mathcal{A}$ admits a coinverse (antipode) $S : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$S(x) = x^{-1} \qquad S(\theta) = -x^{-1}\theta x^{-1}. \tag{15}$$

The coinverse has the properties of an inverse and we have $S^2 = 1$. Indeed,

$$S^{-1}(x) = S(x) \qquad S^{-1}(\theta) = S(\theta).$$

Note that

$$\Delta(x^{-1}) = x^{-1} \otimes x^{-1}.$$

It is not difficult to verify the following properties of costructures:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{16}$$

$$\mu \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mu' \circ (\mathrm{id} \otimes \epsilon) \circ \Delta \tag{17}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \tag{18}$$

where id denotes the identity mapping,

$$\mu : \mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{A} \qquad \mu' : \mathcal{A} \otimes \mathcal{C} \longrightarrow \mathcal{A}$$

are the canonical isomorphisms, defined by

$$\mu(k \otimes u) = ku = \mu'(u \otimes k) \qquad \forall u \in \mathcal{A} \qquad \forall k \in \mathcal{C}$$

and m is the multiplication map

$$m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \qquad m(u \otimes v) = uv. \tag{19}$$

The multiplication in $\mathcal{A} \otimes \mathcal{A}$ follows the rule

$$(A \otimes B)(C \otimes D) = (-1)^{\hat{B}\hat{C}} AC \otimes BD. \tag{20}$$

The coproduct, counit and coinverse which are specified above supply the algebra $\mathcal{A}$ with a graded Hopf algebra structure.

3.2. A Hopf algebra structure on Γ

We first note that consistency of a differential calculus with commutation relations (1) means that the algebra Γ is a graded associative algebra generated by the elements of the set $\{x, \theta, \mathrm{d}x, \mathrm{d}\theta\}$.

Since the algebra Γ is generated by the generators set $\{x, \theta, \mathrm{d}x, \mathrm{d}\theta\}$ we must only describe the actions of comaps on the subset $\{\mathrm{d}x, \mathrm{d}\theta\}$. To denote the coproduct, counit and coinverse which will be defined on the algebra Γ with those of $\mathcal{A}$ may be inadvisable. For this reason, we shall denote them with a different notation. To this end we consider a map $\hat{\Delta}_R : \Gamma \longrightarrow \Gamma \otimes \mathcal{A}$ such that

$$\hat{\Delta}_R \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta. \tag{21}$$

Thus we have

$$\begin{aligned} \hat{\Delta}_R(\mathrm{d}x) &= \mathrm{d}x \otimes x \\ \hat{\Delta}_R(\mathrm{d}\theta) &= \mathrm{d}\theta \otimes x + \mathrm{d}x \otimes \theta. \end{aligned} \tag{22}$$

We now define a map ϕ_R as follows:

$$\phi_R(u_1 \mathrm{d}v_1 + \mathrm{d}v_2 u_2) = \Delta(u_1)\hat{\Delta}_R(\mathrm{d}v_1) + \hat{\Delta}_R(\mathrm{d}v_2)\Delta(u_2). \tag{23}$$

Then it can be checked that the map ϕ_R leaves invariant the relations (11) and (12). One can also check that the following identities are satisfied:

$$(\phi_R \otimes \mathrm{id}) \circ \phi_R = (\mathrm{id} \otimes \Delta) \circ \phi_R \qquad (\mathrm{id} \otimes \epsilon) \circ \phi_R = \mathrm{id}. \tag{24}$$

But we do not have a coproduct for the differential algebra because the map $\hat{\Delta}_R$ does not gives an analogue for the derivation property (5), yet. So we consider another map $\hat{\Delta}_L : \Gamma \longrightarrow \mathcal{A} \otimes \Gamma$ such that

$$\hat{\Delta}_L \circ \mathrm{d} = (\tau \otimes \mathrm{id}) \circ \Delta \tag{25}$$

and a map ϕ_L with again (23) by replacing L with R. Here $\tau : \Gamma \longrightarrow \Gamma$ is the linear map of degree zero which gives $\tau(a) = (-1)^{\hat{a}} a$. The map ϕ_L also leaves invariant the relations (11) and (12), and the following identities are satisfied:

$$(\mathrm{id} \otimes \phi_L) \circ \phi_L = (\Delta \otimes \mathrm{id}) \circ \phi_L \qquad (\epsilon \otimes \mathrm{id}) \circ \phi_L = \mathrm{id}. \tag{26}$$

Let us define the map $\hat{\Delta}$ as

$$\hat{\Delta} = \phi_R + \phi_L \tag{27}$$

which will allow us to define the coproduct of the differential algebra. We denote the restriction of $\hat{\Delta}$ to the algebra $\mathcal{A}$ by Δ and the extension of Δ to the differential algebra Γ by $\hat{\Delta}$:

$$\hat{\Delta}|_{\mathcal{A}} = \Delta \qquad \Delta|_{\Gamma} = \hat{\Delta}. \tag{28}$$

It is possible to interpret the first relation in (28) as the definition of $\hat{\Delta}$ and (27) as the definition of $\hat{\Delta}$ on differentials.

One can see that $\hat{\Delta}$ is a linear map and a homomorphism. In fact, for example,

$$\hat{\Delta}(x\ \mathrm{d}x) = (\phi_R + \phi_L)(x\ \mathrm{d}x) = \Delta(x)(\hat{\Delta}_R + \hat{\Delta}_L)(\mathrm{d}x)$$

and with (28)

$$\Delta(x)\hat{\Delta}(\mathrm{d}x) = \Delta(x)[\Delta(1)(\hat{\Delta}_R + \hat{\Delta}_L)(\mathrm{d}x)].$$

Using the coassociativity of Δ, equation (16), we can also show the coassociativity of $\hat{\Delta}$. So the map $\hat{\Delta}$ is a coproduct for the differential algebra Γ.

Similarly, if we define a counit $\hat{\epsilon}$ for the differential algebra as

$$\hat{\epsilon} \circ \mathrm{d} = \mathrm{d} \circ \epsilon = 0 \tag{29}$$

and

$$\hat{\epsilon}|_{\mathcal{A}} = \epsilon \qquad \epsilon|_{\Gamma} = \hat{\epsilon} \tag{30}$$

one has

$$\hat{\epsilon}(\mathrm{d}x) = 0 \qquad \hat{\epsilon}(\mathrm{d}\theta) = 0 \tag{31}$$

where

$$\hat{\epsilon}(u_1\mathrm{d}v_1 + \mathrm{d}v_2u_2) = \epsilon(u_1)\hat{\epsilon}(\mathrm{d}v_1) + \hat{\epsilon}(\mathrm{d}v_2)\epsilon(u_2). \tag{32}$$

Here we used the fact that $\mathrm{d}(1) = 0$.

The next step is to obtain a coinverse $\hat{S}$. For this, it suffices to define $\hat{S}$ such that

$$\hat{S} \circ \mathrm{d} = \mathrm{d} \circ S \tag{33}$$

and

$$\hat{S}|_{\mathcal{A}} = S \qquad S|_{\Gamma} = \hat{S} \tag{34}$$

where

$$\hat{S}(u_1\mathrm{d}v_1 + \mathrm{d}v_2u_2) = \hat{S}(\mathrm{d}v_1)S(u_1) + S(u_2)\hat{S}(\mathrm{d}v_2). \tag{35}$$

So the action of $\hat{S}$ on the generators $\mathrm{d}x$ and $\mathrm{d}\theta$ is as follows:

$$\begin{aligned} \hat{S}(\mathrm{d}x) &= -x^{-1}\,\mathrm{d}x\,x^{-1} \\ \hat{S}(\mathrm{d}\theta) &= -x^{-1}\,\mathrm{d}\theta\,x^{-1} + 2x^{-1}\,\mathrm{d}x\,x^{-1}\theta x^{-1}. \end{aligned} \tag{36}$$

Note that it is easy to check that $\hat{\epsilon}$ and $\hat{S}$ leave invariant the relations (11) and (12).

Consequently, we can say that the structure $(\Gamma, \hat{\Delta}, \hat{\epsilon}, \hat{S})$ is a graded Hopf algebra.

4. Hopf algebra structure of forms on $\mathcal{A}$

In this section we shall define two forms using the generators of $\mathcal{A}$ and show that the algebra of forms is a graded Hopf algebra.

If we call them w and u then one can define them as follows:

$$w = \mathrm{d}x\,x^{-1} \qquad u = \mathrm{d}\theta\,x^{-1} - \mathrm{d}x\,x^{-1}\theta x^{-1}. \tag{37}$$

We denote the algebra of forms generated by two elements w and u by Ω. The generators of the algebra Ω with the generators of $\mathcal{A}$ satisfy the following rules:

(I)

$$xw = pwx \qquad \theta w = -w\theta + (1-p)ux$$
$$xu = pqux \qquad \theta u = pqu\theta. \tag{38a}$$

(II)

$$xw = swx \qquad \theta w = -qrw\theta$$
$$xu = qux + q(qr-s)w\theta \qquad \theta u = qu\theta. \tag{38b}$$

The commutation rules of the generators of Ω are

(I)

$$w^2 = 0 \qquad wu = uw. \tag{39a}$$

(II)

$$w^2 = 0 \qquad wu = qrs^{-1}uw. \tag{39b}$$

We make the algebra Ω into a graded Hopf algebra with the following costructures: the coproduct $\Delta : \Omega \longrightarrow \Omega \otimes \Omega$ is defined by

$$\Delta(w) = w \otimes 1 + 1 \otimes w \qquad \Delta(u) = u \otimes 1 + 1 \otimes u. \tag{40}$$

The counit $\epsilon : \Omega \longrightarrow \mathcal{C}$ is given by

$$\epsilon(w) = 0 \qquad \epsilon(u) = 0 \tag{41}$$

and the coinverse $S : \Omega \longrightarrow \Omega$ is defined by

$$S(w) = -w \qquad S(u) = -u. \tag{42}$$

One can easily check that (16)–(18) are satisfied. Note that the commutation relations (38) and (39) are compatible with Δ, ϵ and S, in the sense that $\Delta(xw) = p\Delta(wx)$, $\Delta(w^2) = 0$ and so on.

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